Strong Stability with respect to weak limit for a Hyperbolic System arising from Gas Chromatography

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Abstract

We investigate a system related to a particular isothermal gas-solid chromatography process, called "Pressure Swing Adsorption", with two species and instantaneous exchange kinetics. This system presents the particularity to have a linearly degenerate eigenvalue: this allows the velocity of the gaseous mixture to propagate high frequency waves. In case of smooth concentrations with a general isotherm, we prove L^1 stability for concentrations with respect to weak limits of the inlet boundary velocity. Using the Front Tracking Algorithm (FTA), we prove a similar result for concentrations with bounded variation (BV) under some convex assumptions on the isotherms. In both cases we show that high frequency oscillations with large amplitude of the inlet velocity can propagate without affecting the concentrations.

Key words: systems of conservation laws, boundary conditions, BV estimates, entropy solutions, linearly degenerate fields, convex isotherms, Front Tracking Algorithm, waves interaction, geometric optics.

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1 Introduction

"Pressure Swing Adsorption (PSA) is a technology used to separate some species from a gas under pressure according to the molecular characteristics and affinity of the species for an adsorbent material. Special adsorptive materials (e.g. zeolites) are used as a molecular sieve, preferentially adsorbing the undesired gases at high pressure. The process then swings to low pressure to desorb the adsorbent material" (source: Wikipedia).

A typical PSA system involves a cyclic process where a number of connected vessels containing adsorbent material undergo successive pressurization and depressurization steps in order to produce a continuous stream of purified product gas. We focus here on a step of the cyclic process, restricted to isothermal behavior.

As in general fixed bed chromatography, each of the d species ($d \geq 2$) simultaneously exists under two phases, a gaseous and movable one with velocity $u(t,x)$ and concentration $c_i(t,x)$ or a solid (adsorbed) other with concentration $q_i(t,x)$, $1 \leq i \leq d$. We assume that mass exchanges between the mobile and the stationary phases are infinitely fast, thus the two phases are constantly at composition equilibrium: the concentrations in the solid phase are given by some relations $q_i = q_i^*(c_1, ..., c_d)$ where the functions q_i^* are the so-called equilibrium isotherms. A theoretical study of a model with finite exchange kinetics was presented in [5] and a numerical approach was developed in [6].

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In gas chromatography, velocity variations accompany changes in gas composition, especially in the case of high concentration solute: it is known as the sorption effect. In the present model, the sorption effect is taken into account through a constraint on the pressure (or on the density in this isothermal case). See [31] and [35] for a precise description of the process and [10] for a survey on various related models.

The system for two species ($d = 2$) with three unknowns (u, c_1, c_2) is:

$$\begin{aligned} \partial_t(c_1 + q_1^*(c_1, c_2)) + \partial_x(u\, c_1) &= 0, && (1)\\ \partial_t(c_2 + q_2^*(c_1, c_2)) + \partial_x(u\, c_2) &= 0, && (2)\\ c_1 + c_2 &= \rho(t), && (3) \end{aligned}$$

with suitable initial and boundary data. The function ρ represents the *given* total density of the mixture. The experimental device is realized so that it is a given function depending only upon time and in the sequel we assume that $\rho \equiv 1$ (which is not really restrictive from a theoretical point of view). First existence results of large solutions satisfying some entropy criterium in the case of two chemical species were obtained in [8, 9]. In the previous system, it appears that we can expect strong singularities with respect to time for the velocity u. For instance, let $c_1(t, x) \equiv \underline{c}_1$ be a constant, $c_2(t, x) \equiv 1 - \underline{c}_1$, $u(t, x) \equiv u_b(t)$ where u_b is any L^∞ function, then (c_1, c_2, u) is a weak solution of (1),(2),(3). So we can build solutions with a strong oscillating velocity for this system. Furthermore high oscillations of the incoming velocity u_b slightly perturb the concentration as we will see. Notice that we seek positive solutions (c_1, c_2), thus, in view of (3) with $\rho \equiv 1$, c_1, c_2 must satisfy

$$0 \le c_1,\, c_2 \le 1.$$

We use the following notations, introduced in [9]: we set $c = c_1 \in [0, 1]$ and

$$\begin{aligned} q_i(c) &= q_i^*(c, 1 - c), \quad i = 1, 2,\\ h(c) &= q_1(c) + q_2(c),\\ I(c) &= c + q_1(c). \end{aligned}$$

Adding (1) and (2) we get, thanks to (3):

$$\partial_t(q_1(c) + q_2(c)) + \partial_x u = 0,$$

thus our purpose is to study the following system:

$$\left\{ \begin{aligned} \partial_t I(c) + \partial_x(u\, c) &= 0,\\ \partial_t h(c) + \partial_x u &= 0, \end{aligned} \right. \tag{4}$$

supplemented by initial and boundary data:

$$\left\{ \begin{aligned} c(0, x) &= c_0(x) \in [0, 1], \quad x > 0,\\ c(t, 0) &= c_b(t) \in [0, 1], \quad t > 0,\\ u(t, 0) &= u_b(t) > 0, \quad t > 0. \end{aligned} \right. \tag{5}$$

Notice that we assume in (5) an incoming flux at the boundary, i.e. $\forall t > 0,\ u_b(t) > 0$. In the case where the first species is inert, that is $q_1 = 0$, the I function reduces to identity.

System (4) has a null eigenvalue as the system exposed in [4], but, instead [4], we cannot reduce this system to a single equation for general solutions with shocks. In [3] is studied another interesting 2x2 system with a linearly degenerate eigenvalue which modelises some traffic flow. As in [17, 18, 16, 15, 30], the zero eigenvalue makes possible the existence of stratified solutions or the propagation of large-amplitude high frequency waves. Usually, for genuinely nonlinear conservation laws, only high oscillating solutions with small amplitude can propagate: see for instance [22, 14].

In this paper we prove, for large data, that the velocity is a stratified solution in the following sense: $u(t, x) = u_b(t)\, v(t, x)$ where v is as regular as the concentration c and more than the boundary

data u_b. This decomposition for the velocity allows high oscillations with large amplitude for velocity to propagate, without affecting the concentration. For this quasilinear system we have propagation of high oscillations with large amplitude for velocity as in a semilinear system, see for instance [24, 25, 27], and we have strong profile for u with double scale as in [26].

This also permits to pass to the weak limit for u at the boundary and to the strong limit in the interior for the concentration. For the smooth case, we have no restriction on the isotherms, but for the realistic case with shock-waves, we restrict ourselves to the classical treatment of hyperbolic systems: eigenvalues are linearly degenerate or genuinely nonlinear. Furthermore we obtain better interaction estimates when the shock and rarefaction curves are monotonous. It is the case for instance for an inert gas and an active gas with the Langmuir isotherm. We conjecture that our result is still valid for general isotherms with piecewise genuinely nonlinear eigenvalue.

The paper is organized as follows. In Section 2 we recall some basics results from [9] concerning hyperbolicity, entropies, weak entropy solutions of System (4).

In Section 3, we study the case where concentration is smooth and the velocity is only L^∞.

In the remainder of the paper we study the case with only BV concentrations. In short section 4 we briefly expose the Front Tracking Algorithm (FTA) for System (4).

Section 5 is devoted to the study of both shock and rarefaction curves. We state the assumptions that we need to perform estimates with the Front Tracking Algorithm. These assumptions restrict us to convex (or concave) isotherms and we give some examples from chemistry. We obtain the fundamental interaction estimates in Section 6 and BV estimates for v in Section 7. Finally, we obtain strong stability for concentration with respect to weak limit on the boundary velocity in Section 8.

2 Hyperbolicity and entropies

In order the paper to be self contain, we recall without any proof some results exposed in [9].

It is well known that it is possible to analyze the system of Chromatography, and thus System (4), in terms of hyperbolic system of P.D.E. provided we exchange the time and space variables and $u > 0$: see [32] and also [34] for instance. In this framework the vector state will be $U = \begin{pmatrix} u \\ m \end{pmatrix}$ where $m = u\,c$ is the flow rate of the first species. In this vector state, u must be understood as $u\,\rho$, that is the total flow rate.

In the sequel, we will make use of the function $f = q_1\,c_2 - q_2\,c_1$ introduced by Douglas and *al.* in [28], written here under the form

$$f(c) \quad = \quad q_1\,c_2 - q_2\,c_1 = q_1(c) - c\,h(c). \tag{6}$$

Any equilibrium isotherm related to a given species is always increasing with respect to the corresponding concentration (see [28]) i.e. $\dfrac{\partial q_i^*}{\partial c_i} \geq 0$. Since $c = c_1$ and $c_2 = 1 - c$, it follows:

$$q_1' \geq 0 \geq q_2'. \tag{7}$$

Let us define the function H by

$$H(c) \quad = 1 + (1-c)\,q_1'(c) - c\,q_2'(c) \quad = 1 + q_1'(c) - ch'(c). \tag{8}$$

From (7), H satisfies $H \geq 1$ and we have the following relation between f, H and h:

$$f''(c) \quad = \quad H'(c) - h'(c).$$

2.1 Hyperbolicity

Concerning hyperbolicity, we refer to [20, 36, 37]. System (4) takes the form

$$\partial_x U + \partial_t \Phi(U) = 0 \text{ with } U = \begin{pmatrix} u \\ m \end{pmatrix} \text{ and } \Phi(U) = \begin{pmatrix} h(m/u) \\ \\ I(m/u) \end{pmatrix}. \tag{9}$$

The eigenvalues are:

$$0 \quad \text{and} \quad \lambda = \frac{H(c)}{u},$$

thus in view of (8) the system is strictly hyperbolic. The zero eigenvalue is of course linearly degenerate, moreover the right eigenvector $r = \begin{pmatrix} h'(c) \\ 1 + q_1'(c) \end{pmatrix}$ associated to λ satisfies $d\lambda \cdot r = \dfrac{H(c)}{u^2} f''(c)$, thus λ is genuinely nonlinear in each domain where $f'' \neq 0$.

Proposition 2.1 ([9] Riemann invariants)
System (4) admits the two Riemann invariants:

$$c \quad \textit{and} \quad w = \ln(u) + g(c) = L + g(c), \qquad \textit{where } g'(c) = \frac{-h'(c)}{H(c)} \quad \textit{and } L = \ln(u).$$

Furthermore this system can be rewritten for smooth solutions as:

$$\partial_x c + \frac{H(c)}{u} \partial_t c = 0, \qquad \partial_x(\ln(u) + g(c)) = \partial_x w = 0. \tag{10}$$

2.2 Entropies

Dealing with entropies, it is more convenient, as shown in [9], to work with the functions

$$G(c) = \exp(g(c)), \quad W = \exp(w) = u\, G(c).$$

Notice that G is a positive solution of $HG' + h'G = 0$.
Denote $E(c,u)$ any smooth entropy and $Q = Q(c,u)$ any associated entropy flux. Then, for smooth solutions, $\partial_x E + \partial_t Q = 0$. Moreover:

Proposition 2.2 ([9] Representation of all smooth entropies)
The smooth entropy functions for System (4) are given by

$$E(c,u) \quad = \quad \phi(w) + u\, \psi(c)$$

where ϕ and ψ are any smooth real functions. The corresponding entropy fluxes satisfy

$$Q'(c) \quad = \quad h'(c)\, \psi(c) + H(c)\, \psi'(c).$$

Moreover, in [7], the authors looked for convex entropies for System (9) (i.e. System (4) written in the (u,m) variables) in order to get a kinetic formulation. The next proposition gives us a family of degenerate convex entropies independently of convexity of f or of the isotherms.

Proposition 2.3 ([9] Existence of degenerate convex entropies)
If ψ is convex or degenerate convex, i.e. $\psi'' \geq 0$, then $E = u\,\psi(c)$ is a degenerate convex entropy.

There are some few cases (water vapor or ammonia for instance) where the isotherm is convex. There is also the important case with an inert carrier gas and an active gas with a concave or convex isotherm (see [8, 9, 10]). In these cases, the next proposition ensures the existence of λ-Riemann invariants which are also strictly convex entropies. In such cases, w is monotonous with respect to x for any entropy solution.

Proposition 2.4 ([9] When λ-Riemann invariant is a convex entropy)
There are strictly convex entropy of the form $E = \phi(w)$ if and only if G'' does not vanish.
More precisely, for $\alpha > 0$, $E_\alpha(c,u) = u^\alpha\, G^\alpha(c)$ is an increasing entropy with respect to the Riemann invariant W. It is strictly convex for $\alpha > 1$ if $G'' > 0$ and for $\alpha < 1$ if $G'' < 0$.

Unfortunately, when G has an inflexion point such system does not admit any strictly convex entropy. When one gas is inert, it is always the case if the sign of the second derivative of the isotherm changes. See for instance [9] for the BET isotherm.

Remark 2.1 *In general, System (4) is not in the Temple class. It is the case if and only if f'' does not vanish and $\partial_x W = 0$ for all entropy solution ([11]). For instance, System (4) with two linear isotherms is in the Temple class.*

Proposition 2.5 ([9] Non Existence of strictly convex entropy)
If sign of G'' changes then System (4) does not admit strictly convex smooth entropy.

2.3 Definition of weak entropy solution

We have seen that there are two families of entropies: $u\,\psi(c)$ and $\phi(u\,G(c))$.
The first family is degenerate convex (in variables (u, uc)) provided $\psi'' \geq 0$. So, we seek after weak entropy solutions which satisfy $\partial_x\,(u\,\psi(c)) + \partial_t Q(c) \leq 0$ in the distribution sense.
The second family is not always convex. There are only two interesting cases, namely $\pm G''(c) > 0$ for all $c \in [0,1]$. When $G'' > 0$ and $\alpha > 1$, we expect to have $\partial_x(u\,G(c))^\alpha \leq 0$ from Proposition 2.4. But, the mapping $W \mapsto W^\alpha$ is increasing on $\mathbb{R}^+$. So, the last inequality reduces to $\partial_x(u\,G(c)) \leq 0$. In the same way, if $G'' < 0$, we get $\partial_x(u\,G(c)) \geq 0$.
Now, we can state a mathematical definition of weak entropy solutions.

Definition 2.1 *Let be $T > 0$, $X > 0$, $u \in L^\infty((0,T)\times(0,X), \mathbb{R}^+)$, $0 \leq c(t,x) \leq \rho \equiv 1$ for almost all $(t,x) \in (0,T)\times(0,X)$. Then (c,u) is a* **weak entropy solution** *of System (4)-(5) with respect to the family of entropies $u\,\psi(c)$ if, for all convex (or degenerate convex) ψ:*

$$\frac{\partial}{\partial x}\,(u\,\psi(c)) + \frac{\partial}{\partial t} Q(c) \quad \leq \quad 0, \tag{11}$$

in $\mathcal{D}'([0,T[\times[0,X[)$, where $Q' = H\psi' + h'\psi$, that is, for all $\phi \in \mathcal{D}([0,T[\times[0,X[)$:

$$\int_0^X \int_0^T \left(u\,\psi(c)\,\partial_x\phi + Q(c)\,\partial_t\phi\right)\,dt\,dx + \int_0^T u_b(t)\,\psi(c_b(t))\,\phi(t,0)\,dt + \int_0^X Q(c_0(x))\,\phi(0,x)\,dx \geq 0.$$

Remark 2.2 *If $\pm G'' \geq 0$ then $u\,\psi = \pm u\,G(c)$ is a degenerate convex entropy, with entropy flux $Q \equiv 0$, contained in the family of entropies $u\,\psi(c)$. So, if G'' keeps a constant sign on $[0,1]$, (c,u) has to satisfy:*

$$\pm\,\frac{\partial}{\partial x}\,(u\,G(c)) \leq 0, \qquad \text{if } \pm\,G'' \geq 0 \text{ on } [0,1]. \tag{12}$$

Notice that the entropies $u\,\psi(c)$ and the entropy $u\,G(c)$ are linear with respect to the velocity u.

2.4 About the Riemann Problem

The implementation of the Front Tracking Algorithm used extensively from Section 4 requires some results about the solvability of the following Riemann problem:

$$\left\{\begin{array}{rcl} \partial_x u + \partial_t h(c) & = & 0, \\ \partial_x(uc) + \partial_t I(c) & = & 0, \end{array}\right. \tag{13}$$

$$c(0,x) = c^- \in [0,1], \quad x > 0, \qquad \left\{\begin{array}{rcl} c(t,0) & = & c^+ \in [0,1], \\ u(t,0) & = & u^+ > 0, \end{array}\right. \quad t > 0. \tag{14}$$

We are classically looking for a selfsimilar solution, i.e.: $c(t,x) = C(z)$, $u(t,x) = U(z)$ with $z = \dfrac{t}{x} > 0$. The answer is given by the three following results ([9]).

Proposition 2.6 *Assume for instance that* $0 \leq a < c^- < c^+ < b \leq 1$ *and* $f'' > 0$ *in* $]a, b[$. *Then the only smooth self-similar solution of (13)-(14) is such that :*

$$\left\{ \begin{array}{rcll} C(z) & = & c_-, & 0 < z < z_-, \\ \dfrac{dC}{dz} & = & \dfrac{H(C)}{z\, f''(C)}, & z_- < z < z_+, \\ C(z) & = & c_+, & z_+ < z, \end{array} \right. \tag{15}$$

where $z^+ = \dfrac{H(c^+)}{u^+}$, $z^- = z^+ \, e^{-\Phi(c^+)}$ *with* $\Phi(c) = \displaystyle\int_{c^-}^{c} \frac{f''(\xi)}{H(\xi)}\, d\xi$. *Moreover* $u^- = \dfrac{H(c^-)}{z^-}$ *and* U *is given by:*

$$\left\{ \begin{array}{rcll} U(z) & = & u_-, & 0 < z < z_-, \\ U(z) & = & \dfrac{H(C(z))}{z}, & z_- < z < z_+, \\ U(z) & = & u_+ & z_+ < z. \end{array} \right. \tag{16}$$

Proposition 2.7 *If* (c^-, c^+) *satisfies the following admissibility condition equivalent to the Liu entropy-condition ([29]):*

$$\text{for all } c \text{ between } c^- \text{ and } c^+, \quad \frac{f(c^+) - f(c^-)}{c^+ - c^-} \leq \frac{f(c) - f(c^-)}{c - c^-},$$

then the Riemann problem (13)-(14) is solved by a shock wave defined as:

$$C(z) = \left\{ \begin{array}{lll} c^- & \text{if} & 0 < z < s, \\ c^+ & \text{if} & s < z \end{array} \right. , \qquad U(z) = \left\{ \begin{array}{lll} u^- & \text{if} & 0 < z < s, \\ u^+ & \text{if} & s < z, \end{array} \right. \tag{17}$$

where u^- *and the speed* s *of the shock are obtained through*

$$\frac{[f]}{u^- \, [c]} + \frac{1 + h^-}{u^-} = s = \frac{[f]}{u^+ \, [c]} + \frac{1 + h^+}{u^+},$$

where $[c] = c^+ - c^-$, $[f] = f^+ - f^- = f(c^+) - f(c^-)$, $h^+ = h(c^+)$, $h^- = h(c^-)$.

Proposition 2.8 *Two states* U^- *and* U^+ *are connected by a contact discontinuity if and only if* $c^- = c^+$ *(with of course* $u^- \neq u^+$*), or* $c^- \neq c^+$ *and* f *is affine between* c^- *and* c^+.

It appears from these results that we can build a weak entropy solution of the Riemann problem (13)-(14) in a very simple way (see [9]), similar to the scalar case with flux f, for any data. In particular, if f'' has a constant sign (which is the framework in Section 4), the Riemann problem is always solved by a simple wave.

3 Case with smooth concentration

System (4) has the strong property that there exist weak entropy solutions with *smooth* concentration c on $(0, T) \times (0, X)$ but not necessarily smooth velocity u, for some positive constants T and X. Furthermore, c is the solution of a scalar conservation law.

3.1 Existence of weak entropy solutions with smooth concentration

For this section, we refer to [16], [15]. We have a similar result in [8] but only with smooth velocity. Here, we obtain by the classical method of characteristics *existence and uniqueness* of a weak entropy solution with smooth concentration and only L^∞ velocity.

Theorem 3.1 (Unique weak entropy solution with *smooth* concentration)
*Let $T_0 > 0$, $X > 0$, $c_0 \in W^{1,\infty}([0,X],[0,1])$, $c_b \in W^{1,\infty}([0,T_0],[0,1])$, $\ln u_b \in L^\infty([0,T_0],\mathbb{R})$.
If $c_0(0) = c_b(0)$ then there exists $T \in]0,T_0]$ such that System (4)-(5) admits a* **unique** *weak entropy solution (c,u) on $[0,T]\times[0,X]$ with*

$$c \in W^{1,\infty}([0,T]\times[0,X],[0,1]), \qquad \ln u \in L^\infty([0,T], W^{1,\infty}([0,X],\mathbb{R})).$$

Furthermore, for any $\psi \in C^1([0,1],\mathbb{R})$, setting

$$F'(c) = (H(c)\,G(c))^{-1} \text{ and } Q' = H\,\psi' + h'\,\psi,$$

(c,u) satisfies:

$$\partial_x(u\,\psi(c)) + \partial_t Q(c) = 0, \quad \partial_x(u\,G(c)) = 0, \tag{18}$$

$$\partial_t c + u_b(t)\,G(c_b(t))\,\partial_x F(c) = 0. \tag{19}$$

Proof: we build a solution using the Riemann invariants and we check that such a solution is an entropy solution. Next, we prove uniqueness.
Using the Riemann invariant $W = u\,G(c)$ ($\partial_x W = 0$) and the boundary data we define u by:

$$u(t,x) = \frac{u_b(t)\,G(c_b(t))}{G(c(t,x))},$$

so u is smooth with respect to x. Then, the first equation of (10) can be rewritten as follows:

$$\partial_t c + \mu\,\partial_x c = 0, \qquad \text{with} \qquad \mu = \lambda^{-1} = \frac{u}{H(c)} = \frac{u_b(t)\,G(c_b(t))}{H(c)\,G(c)} = \mu(t,c). \tag{20}$$

We solve (20) supplemented by initial-boundary value data (c_0, c_b) by the standard characteristics method. Let us define, for a given (τ,x), $X(\cdot,\tau,x)$ as the solution of:

$$\frac{dX(s,\tau,x)}{ds} = \mu(s, c(s, X(s,\tau,x))), \qquad X(\tau,\tau,x) = x.$$

Since $\dfrac{dc}{ds}(s, X(s,\tau,x)) = 0$ from (20), we have

$$X(s,\tau,x) = x - b(s,\tau)\,F'(c(\tau,x)) \quad \text{with} \quad b(s,\tau) = \int_s^\tau u_b(\sigma)\,G(c_b(\sigma))\,d\sigma.$$

Now, for some $T \in [0,T_0]$ defined later on, we split $\Omega = [0,T]\times[0,X]$ according to the characteristic line Γ issuing from the corner $(0,0)$, i.e. we define the sets $\Omega^\pm = \{(t,x) \in \Omega,\ \pm(x - X(t,0,0)) \geq 0\}$. Since $\partial_x X(t,0,x) = 1 - b(t,0)\,F''(c_0(x))\,\partial_x c_0(x)$, $b(0,0) = 0$ and $b(.,0) \in W^{1,\infty}(\Omega^+)$, the mapping $x \mapsto X(t,0,x)$ is a Lipschitz diffeomorphism for $0 \leq t \leq T$ with $T \in]0,T_0]$ small enough. Then we define on Ω^+, for each $t \in [0,T]$, $\xi(t,x)$ such as $X(t,0,\xi(t,x)) = x$. Then we have $c(t,x) = c_0(\xi(t,x))$ on Ω^+. Furthermore $\partial_t \xi = -\partial_s X/\partial_x X$ and thus c is Lipschitz continuous in time and space on Ω^+.
We work in a similar way on Ω^- and $c \in W^{1,\infty}(\Omega^-)$. Since c is continuous on Γ from the compatibility conditions $c_0(0) = c_b(0)$ we have $c \in W^{1,\infty}(\Omega)$.
By construction (c,u) satisfies (10) rewritten as follows:

$$\partial_x \ln u = -\partial_x g(c), \qquad u\,\partial_x c + H\,\partial_t c = 0.$$

These equations imply:

$$\partial_x u = -u\,\partial_x g(c) = -u\,g'(c)\,\partial_x c = -u\,g'(c)\left(-\frac{H(c)}{u}\,\partial_t c\right) = -h'(c)\,\partial_t c = -\partial_t h(c).$$

Now we check that (c,u) satisfies (18). Let ψ be a C^1 function. Using the identity $Q' = h'\psi + H\psi'$ and the previous equations we have:

$$\begin{aligned}\partial_x(u\,\psi(c)) + \partial_t Q(c) &= \psi\,\partial_x u + u\,\psi'\,\partial_x c + Q'\,\partial_t c = \psi\,(\partial_x u + h'\,\partial_t c) + \psi'\,(u\partial_x c + H\,\partial_t c)\\ &= \psi \times 0 + \psi' \times 0 = 0.\end{aligned}$$

By the way (18) implies (11), so (c,u) is an entropy solution of System (4).
We now prove the uniqueness of such a weak entropy solution.
Precisely, if $c \in W^{1,\infty}([0,T]\times[0,X],[0,1])$ and $\ln u \in L^\infty((0,T),W^{1,\infty}(0,X))$ satisfy (11) in $\mathcal{D}'([0,T[\times[0,X[)$ with initial-boundary data c_0, c_b, u_b then we show that (c,u) is necessarily the previous solution built by the method of characteristics.
Choosing the convex functions $\psi(c) = \pm 1$ and $\psi(c) = \pm c$ we obtain (4). The main ingredient to conclude the proof is the fact that u admits a classical partial derivative only with respect to x. Thus classical computations with smooth functions to obtain (10) as in the proof of Proposition 2.1 are still valid. Now (c,u) satisfies (10), which implies from the beginning of the proof of Theorem 3.1 that (c,u) is our previous solution. $\square$

Remark 3.1

1. *Notice that T, X are only depending on $\|\ln(u_b)\|_{L^\infty}$, $\|c_b\|_{W^{1,\infty}}$, $\|c_0\|_{W^{1,\infty}}$. Thus, if $(u_b^\varepsilon)_{0<\varepsilon\le 1}$ is a sequence of boundary velocity data such that $(\ln u_b^\varepsilon)$ is uniformly bounded in $L^\infty(0,T_0)$, and if $(c_0^\varepsilon), (c_b^\varepsilon)$ are some initial and boundary concentration data uniformly bounded in $W^{1,\infty}$ with the compatibility condition at the corner $c_0(0) = c_b(0)$, then there exist $T > 0$ and $X > 0$ and Lipschitz bounds for $c^\varepsilon, \ln u^\varepsilon$ on $[0,T]\times[0,X]$ independent of ε.*
2. *As in [8], we have a global solution with smooth concentration if λ is genuinely nonlinear (for instance an inert case and a Langmuir isotherm), with monotonicity assumptions on c_0 and c_b.*

3.2 Strong stability with respect to velocity

In case of a Lipschitz continuous concentration, we now give a strong stability result for the concentration with respect to a weak limit of the boundary velocity.

Theorem 3.2 (Strong stability for *smooth* concentration)
Let be $T_0 > 0$, $X > 0$, $c_0 \in W^{1,\infty}([0,X],[0,1])$, $c_b \in W^{1,\infty}([0,T_0],[0,1])$ such that $c_0(0) = c_b(0)$, and $(\ln u_b^\varepsilon)_{0<\varepsilon\le 1}$ a bounded sequence in $L^\infty(0,T_0)$. Then, there exists $T \in]0,T_0[$ such that System (4) admits a unique weak entropy solution $(c^\varepsilon, u^\varepsilon)$ with $c^\varepsilon \in W^{1,\infty}([0,T]\times[0,X],[0,1]), \ln u^\varepsilon \in L^\infty([0,T],W^{1,\infty}([0,X],\mathbb{R}))$ and with initial and boundary values:

$$\left\{\begin{array}{rcll} c^\varepsilon(0,x) &=& c_0(x) \in [0,1], & x > 0,\\ c^\varepsilon(t,0) &=& c_b(t) \in [0,1], & t > 0,\\ u^\varepsilon(t,0) &=& u_b^\varepsilon(t) > 0, & t > 0. \end{array}\right. \tag{21}$$

If (u_b^ε) converges towards $\overline{u}_b$ in $L^\infty(0,T_0)$ weak$-*$ when ε goes to 0, then (c^ε) converges in $L^\infty([0,T]\times[0,X])$ towards the unique smooth solution of

$$\partial_t c + \overline{u}_b(t)\,G(c_b(t))\,\partial_x F(c) = 0, \qquad c(t,0) = c_b(t),\ c(0,x) = c_0(x). \tag{22}$$

Furthermore we have:

$$\lim_{\varepsilon\to 0}\left\| u^\varepsilon(t,x) - u_b^\varepsilon(t)\frac{G(c_b(t))}{G(c(t,x))}\right\|_{L^\infty([0,T]\times[0,X])} = 0.$$

Proof: thanks to Theorem 3.1, there exists $T > 0$ such that System (4), with initial and boundary values (21) admits the unique weak entropy solution $(c^\varepsilon, u^\varepsilon)$ with smooth concentration in the

previous sense on $[0,T]\times[0,X]$.
Since (c^ε) is bounded in $W^{1,\infty}$, up to a subsequence, (c^ε) converges strongly in L^∞ to c. Using (19) in conservative form, we can pass to the limit and get (22). Problem (22) has a unique solution by the method of characteristics. Thus, the whole sequence (c^ε) converges. We recover the last limit for u^ε thanks to $\partial_x(u\,G(c))=0$. $\square$

Notice that if $\overline{u}_b$ is a constant function for instance $u_b^\varepsilon(t)=u_b(t/\varepsilon)$ with u_b periodic we can compute the concentration with only using a constant velocity (the mean velocity) as in liquid chromatography.
An example from geometric optics: if $u_b^\varepsilon(t) = u_b\left(t,\dfrac{t}{\varepsilon}\right)$ where $u_b(t,\theta)\in L^\infty((0,T),C^0(\mathbb{R}/\mathbb{Z}))$, $\inf u_b>0$, we have a similar result with Equation (22) for c where $\overline{u}_b(t)=\displaystyle\int_0^1 u_b(t,\theta)\,d\theta$ and a profile U:

$$\lim_{\varepsilon\to 0}\left\| u^\varepsilon(t,x) - U\left(t,x,\frac{t}{\varepsilon}\right)\right\|_{L^\infty} = 0 \quad \text{where } U(t,x,\theta) = u_b(t,\theta)\frac{G(c_b(t))}{G(c(t,x))}.$$

4 Front Tracking Algorithm

In Section 3, where c is smooth and $\ln u_b$ is in $L^\infty(0,T)$, we have seen that there exists a *smooth* function v such that

$$u(t,x) \quad = \quad u_b(t)\,v(t,x). \tag{23}$$

Furthermore c satisfies the *scalar* conservation law (19). For only BV data we cannot expect to obtain such a scalar conservation law for the concentration, except in the case of linear isotherms. In that case, the scalar conservation law (19) and System (4) have the same solution for the Riemann Problem, but linear isotherms are of a poor interest from Chemical Engineering point of view. The first interesting case is the case with an inert gas and a Langmuir isotherm, first mathematically studied in [8].

Nevertheless we guess that (23) is still true with $v\in BV$. From [8, 9] we have yet obtained BV regularity with respect to x with a Godunov scheme. To get BV regularity with respect to t we will use a more precise algorithm to study wave interactions, namely a Front Tracking Algorithm (FTA).

The Front Tracking method for scalar conservation laws was introduced by Dafermos, [19]. The method was extended to genuinely nonlinear systems of two conservation laws by DiPerna [21]. For our purpose, we do not use the generalisation to genuinely nonlinear systems of any size by Bressan [12] or Risebro [33].

The FTA is much more complicated when an eigenvalue is piecewise genuinely nonlinear, see [2, 1, 23]. Then, we restrict ourselves to the case where λ is genuinely nonlinear, which allows us to treat some relevant cases from the point of view of chemical engineering like an inert gas with a Langmuir isotherm, two active gas with a binary Langmuir isotherm for instance. For this purpose we work in the framework exposed in the recent and yet classical Bressan's Book [13]. In this framework we assume $f''\geq 0$, then a Riemann problem presents only two waves:

1. a contact discontinuity with speed 0,
2. a rarefaction wave with speed $\lambda>0$
   or a shock wave with speed between λ^- and λ^+, characteristic speeds associated to the left and right states, respectively.

Let be $\delta>0$. A δ-approximate Front Tracking solution of System (4) is a pair of piecewise constant function $c^\delta(t,x), u^\delta(t,x)$, whose jumps are located along finitely many straight lines $t=t_\alpha(x)$ in the $t-x$ plane and approximately satisfy the entropy conditions. For each $x>0$ and $\psi''\geq 0$, one should thus have an estimate of the form:

$$\sum_\alpha \left([u^\delta\,\psi(c^\delta)] - \frac{dt_\alpha}{dx}[Q(c^\delta)]\right)(t_\alpha,x) \quad \leq \quad \mathcal{O}(\delta), \tag{24}$$

where $[u] = u^+ - u^-$ is the jump across a jump line, and the sum is taken over all jump for x fixed. Inequality (24) implies that (c^δ, u^δ) is "almost an entropy solution":

$$\partial_x u^\delta \psi(c^\delta) + \partial_t \psi(c^\delta) \leq \mathcal{O}(\delta). \tag{25}$$

That's enough to get an entropy solution "issued from FTA" when δ goes to zero.

Since we want to only use piecewise constant functions, it is convenient to approximate a continuous rarefaction wave by a piecewise constant function. For this purpose, the rarefaction curve is dicretized with a step of order δ and then (24) still holds.

We now briefly describe an algorithm which generates these Front Tracking approximations. The construction starts on the initial line $x = 0$ and the boundary $t = 0$ by taking a piecewise constant approximation of initial value $c_b(t), u_b(t)$ and boundary values $c_0(x)$. Let $t_1 < \cdots < t_N$, $\tilde{x}_1 < \cdots < \tilde{x}_M$ be the points where initial-boundary values are discontinuous. For each $\alpha = 1, \cdots, N$, the Riemann problem generated by the jump of initial constant values at $(t_\alpha, x = 0)$ is approximately solved on a forward neighborhood of $(t_\alpha, 0)$ in the $t - x$ plane by a function invariant on line $t - t_\alpha = \mathrm{a}\, x$, for all positive a, and piecewise constant. Notice that the boundary is characteristic, then we have only one wave associated with the speed λ in the corner $(0, 0)$.

The approximate solution (c^δ, u^δ) can then be prolonged until $x_1 > 0$ is reached, when the first set of interactions between two wave-fronts takes place. If $x_1 > \tilde{x}_1$ we first have to solve the characteristic boundary Riemann problem at $(t = 0, x = \tilde{x}_1)$. Since $(c^\delta, u^\delta)(., x_1)$ is still a piecewise constant function, the corresponding Riemann problems can again be approximately solved within the class of piecewise constant functions. The solution is then continued up to a value x_2 where the next characteristic boundary Riemann problem occurs or the second set of wave interactions takes place, and so on.

According to this algorithm, contact discontinuity fronts travel with speed zero, shock fronts travel exactly with Rankine-Hugoniot speed, while rarefaction fronts travel with an approximate characteristic speed. However, one exception to this rule must be allowed if three or more fronts meet at the same point. To avoid this situation, we must change the positive speed λ of one of the incoming shock fronts or rarefaction fronts. Of course this change of speed can be chosen arbitrarily small and we have again Inequality (24).

Notice that, for 2×2 system the number of wave-fronts cannot approach infinity in finite $x > 0$. DiPerna shows in [21] that the process of regenerating the solution by solving local Riemann problems yields an approximating solution within the class of piecewise constant functions that is globally defined and that contains only a finite number of discontinuities in any compact subset of the $t - x$ quarter plane $t \geq 0, x \geq 0$. We then do not consider non-physical fronts as in [13] for general $n \times n$ systems with $n \geq 3$.

5 About the shock and rarefaction curves

In this section we state our assumptions to use the FTA with large data. Precisely we work in classical hyperbolic case, namely, eigenvalues are linearly degenerate or genuinely nonlinear. We assume:

$$\lambda = \frac{H(c)}{u} \text{ is } \textbf{genuinely nonlinear} \qquad \text{i.e } f \text{ is } \textbf{convex} \text{ on } [0, 1]. \tag{26}$$

Actually λ is genuinely nonlinear for $f'' \neq 0$, but since $f = c_1 q_2 - c_2 q_1$ (see (6)) we can assume that $f'' > 0$ exchanging the gas labels 1 and 2 if necessary.
Our analysis of wave interactions in Section 6 is more precise with monotonous λ-wave curves, then we also assume:

$$\lambda\text{-wave curves are } \textbf{monotonous}. \tag{27}$$

To state precisely this last assumption let us introduce some notations. Let (c_-, L_-) be a left constant state connected to (c_+, L_+) a right constant state by a λ-wave curve. In the genuinely linear case, with Assumption (26), λ-wave curve is a rarefaction curve with $c_- < c_+$ or a shock

curve with $c_- > c_+$. The sign of $[c] = c_+ - c_-$ comes from the general study of the Riemann problem in [9]. From the Riemann invariant $w = \ln u + g(c)$ and the Rankine-Hugoniot conditions a λ-wave curve can be written as follows (see [9]):

$$[L] = L_+ - L_- = \ln u_+ - \ln u_- \quad = \quad T(c_+, c_-) = \begin{cases} -[g] = -(g(c_+) - g(c_-)) & \text{if } c_- < c_+ \\ S(c_+, c_-) & \text{else} \end{cases} \tag{28}$$

We give an explicit formula for S in Lemma 5.1.
Notice that we use only one Riemann invariant, namely c, to write λ-wave curves. Indeed $L = \ln u$ and c have quite different behavior as seen in [8, 9] and this paper. Furthermore we can give some simple criterion to have monotonous λ-wave curves. For instance, as $g' = -h'/H$, the rarefaction curve is monotonous if and only if h is monotonous. A chemical example, investigated in [8], is the case of an inert gas ($q_1 = 0$) and an active gas with a Langmuir isotherm: $q_2^*(c_2) = Q_2 \dfrac{K_2 c_2}{1 + K_2 c_2}$.
For this case we have

$$f'' > 0, \qquad h' < 0, \qquad \frac{\partial S}{\partial c_-} \geq 0 \geq \frac{\partial S}{\partial c_+}. \tag{29}$$

The first condition of (29) gives us (26) and the last one gives us (27).
Notice that if we exchange labels 1 and 2 for gas, Inequalities (29) simply become:

$$f'' < 0 < h', \quad \frac{\partial S}{\partial c_-} \leq 0 \leq \frac{\partial S}{\partial c_+}.$$

Let us give some isotherm examples such that (29) is satisfied.

Proposition 5.1 *For the following examples, Assumptions (26), (27) are valid:*

1. *one gas is inert: $q_1 = 0$, and the other has a concave isotherm: $q_2^{''} \leq 0$,*
2. *two active gas with linear isotherms: $q_i^*(c_1, c_2) = K_i c_i$, $i = 1, 2$,*
3. *two active gas with binary Langmuir isotherms: $q_i^*(c_1, c_2) = \dfrac{Q_i K_i c_i}{1 + K_1 c_1 + K_2 c_2}$, $i = 1, 2$, where positive constants $Q_1, Q_2, K_1 \geq K_2$ satisfy: $Q_1 K_1 < Q_2 K_2$.*

Furthermore, for two active gas with binary Langmuir isotherms, λ is genuinely nonlinear, i.e. (26) is satisfied, if $Q_1 K_1 \neq Q_2 K_2$.

The first case is the most classical case when only one gas is active and his isotherm has no inflexion point, for instance the Langmuir isotherm.
The second case is less interesting in chemistry and only valid when concentrations are near constant states.
For the third case, notice that $K_1 \geq K_2$ is not really an assumption (exchange the labels if necessary).

***Proof of Proposition* 5.1:** we use some technical Lemmas postponed to Subsection 5.1. The point is to satisfy (29).

1. Case with an inert gas: we have $h = q_2$, $f(c) = -c\,h(c)$, $f' = -h - ch'$, $f'' = -2h' - c\,h''$, which implies $h' = q_2' \leq 0$, $h'' = q_2" \leq 0$ and then $f'' \geq 0$. We conclude thanks to Lemmas 5.3 and 5.4.

2. Case with linear isotherms: linear isotherms are $q_1(c) = K_1 c$, $q_2(c) = K_2(1 - c)$ with $K_1 \geq 0$, $K_2 \geq 0$ then $q_1'(c) = K_1 \geq 0$, $q_2'(c) = -K_2 \leq 0$, $h'(c) = q_1'(c) + q_2'(c) = K_1 - K_2$, $f''(c) = 2(K_2 - K_1)$. We assume $K_1 \leq K_2$, then we have $h' \leq 0 \leq f''$. Since $q_i" = 0$, $i = 1, 2$, we conclude thanks to Lemmas 5.3 and 5.5.

3. Case with a binary Langmuir isotherm: we have $q_1(c) = \dfrac{Q_1 K_1 c}{D}$, $q_2(c) = \dfrac{Q_2 K_2 (1 - c)}{D}$ where $D = 1 + K_1 c + K_2(1 - c)$. Then $q_1'(c) = \dfrac{Q_1 K_1 (1 + K_2)}{D^2} \geq 0$, $q_2'(c) = -\dfrac{Q_2 K_2 (1 + K_1)}{D^2} \leq 0$,

$h'(c) = q_1'(c) + q_2'(c) \leq 0$ if and only if $Q_1K_1(1+K_2) \leq Q_2K_2(1+K_1)$,
$q_1''(c) = \dfrac{2Q_1K_1(1+K_2)(K_2-K_1)}{D^3} \leq 0$ if and only if $K_1 \geq K_2$,
$q_2''(c) = \dfrac{2Q_2K_2(1+K_1)(K_1-K_2)}{D^3} \geq 0$ if and only if $K_1 \geq K_2$,
$f''(c) = \dfrac{2(Q_2K_2-Q_1K_1)(1+K_1)(1+K_2)}{D^3} \geq 0$ if and only if $Q_2K_2 \geq Q_1K_1$.
Since $Q_1K_1 \leq Q_2K_2$, we get $f" \geq 0$ and $\dfrac{Q_1}{Q_2} \leq \dfrac{K_2}{K_1}$. $1 \leq \dfrac{1+K_1}{1+K_2}$ because $K_1 \geq K_2$, so we have $\dfrac{Q_1}{Q_2} \leq \dfrac{K_2}{K_1}\dfrac{1+K_1}{1+K_2}$, i.e. $h' \leq 0$. Now we conclude with Lemmas 5.3 and 5.5. □

5.1 Technical lemmas about shock curves

We express the shock curves as follows.

Lemma 5.1 *We have* $\exp\{S(c_+,c_-)\} = \dfrac{u_-}{u_+} = \dfrac{\alpha+h_-}{\alpha+h_+}$, *where* $h_\pm = h(c_\pm)$ *and* $\alpha = \dfrac{[f]}{[c]} + 1$.

Proof: first, from the Rankine Hugoniot conditions: $\dfrac{[uc]}{[c+q_1(c)]} = \dfrac{[u]}{[h]}$, i.e. $[h] = \dfrac{[u][c+q_1(c)]}{[uc]}$, we obtain

$$\frac{u_+}{u_-} = \frac{[c+q_1(c)] - c_-[h]}{[c+q_1(c)] - c_+[h]} \tag{30}$$

where $[c] = c_+ - c_-$ and $[h] = h(c_+) - h(c_-) = h_+ - h_-$, and we get (30) thanks to the following computations:

$$\begin{aligned} [c+q_1(c)] - c_-[h] &= [c+q_1(c)] - c_-\frac{[u][c+q_1(c)]}{[uc]} = \frac{[c+q_1(c)]}{[uc]}([uc] - c_-[u]) \\ &= \frac{[c]u_+}{[uc]}[c+q_1(c)], \end{aligned}$$

$$\begin{aligned} [c+q_1(c)] - c_+[h] &= [c+q_1(c)] - c_+\frac{[u][c+q_1(c)]}{[uc]} = \frac{[c+q_1(c)]}{[uc]}([uc] - c_+[u]) \\ &= \frac{[c]u_-}{[uc]}[c+q_1(c)]. \end{aligned}$$

Rewriting (30) we get

$$\begin{aligned} \frac{u_-}{u_+} &= \frac{[c+q_1(c)] - c_+[h]}{[c+q_1(c)] - c_-[h]} = \frac{[q_1] + [c] - c_+[h]}{[q_1] + [c] - c_-[h]} = \frac{[q_1] + [c] + c_+(h_- - h_+)}{[q_1] + [c] + c_-(h_- - h_+)} \\ &= \frac{[q_1] - c_+h_+ + [c] + h_-c_+}{[q_1] + c_-h_- + [c] - h_+c_-} = \frac{[f] + [c] + h_-[c]}{[f] + [c] + h_+[c]} = \frac{\alpha + h_-}{\alpha + h_+}, \end{aligned}$$

which concludes the proof. □

We need to know the sign of $\alpha + h_\pm$ before studying the sign of $\dfrac{\partial S}{\partial c_\pm}$.

Lemma 5.2 *If* $h' \leq 0$ *and* $c_+ < c < c_-$ *then* $\alpha + h(c_+) \geq \alpha + h(c) \geq \alpha + h(c_-) > 0$.

Proof: since $h' \le 0$ and $c_+ < c_-$ we have $h(c_+) \ge h(c_-)$ and it is enough to show that $\dfrac{[f]}{[c]} + 1 + h(c_-) > 0$. This inequality is equivalent to $[f] + [c] + [c]h(c_-) < 0$ because $[c] = c_+ - c_- < 0$. Since $f(c) = q_1(c) - ch(c)$ the inequality is equivalent to $[q_1] + [c] < c_+[h]$. We know that $q_1' \ge 0$, $c_+ < c_-$, $h' \le 0$ then $[q_1] \le 0$, $[c] < 0$, $[h] \ge 0$ and then $[q_1] + [c] < 0 < c_+[h]$. □

Lemma 5.3 *If $h' \le 0$, if f is convex and if $c_+ < c_-$ then we have $\dfrac{\partial S}{\partial c_+}(c_+, c_-) \le 0$.*

Proof: we have $S(c_+, c_-) = [L] = \ln(u_+) - \ln(u_-) = \ln(\dfrac{u_+}{u_-})$ and $\dfrac{\partial}{\partial c_+}\dfrac{u_+}{u_-} = \dfrac{\partial}{\partial c_+}\dfrac{\alpha + h_+}{\alpha + h_-}$ thanks to Lemma 5.1. A calculus gives $\dfrac{\partial}{\partial c_+}\dfrac{\alpha + h_+}{\alpha + h_-} = \dfrac{1}{(\alpha + h_-)^2}\left(-\dfrac{\partial \alpha}{\partial c_+}[h] + h'(c_+)(\alpha + h_-)\right)$. Now $\dfrac{\partial \alpha}{\partial c_+} \ge 0$ because f is convex, next $[h] \ge 0$ since $h' \le 0$ and $c_+ < c_-$. Lastly $\alpha + h_- > 0$ from Lemma 5.2 and we get $\dfrac{\partial S}{\partial c_+}(c_+, c_-) \le 0$. □

The following result concerns the case with an inert gas:

Lemma 5.4 *If $q_1 = 0$ and $q_2'' \le 0$ then $\dfrac{\partial S}{\partial c_-} \ge 0$ for $c_- > c_+$.*

Proof: if $q_1 = 0$ then $f(c) = -ch(c)$, $h(c) = q_2(c)$ then $h'(c) = q_2'(c) \le 0$. By a direct computation and thanks to Lemma 5.1, we have

$$\frac{u_+}{u_-} = \frac{[c] - c_-[h]}{[c] - c_+[h]} = \frac{[c] - c_+[h] + [c][h]}{[c] - c_+[h]} = 1 + \frac{1}{\dfrac{1}{[h]} - \dfrac{c_+}{[c]}}.$$

But $\dfrac{\partial}{\partial c_-}\dfrac{1}{[h]} < 0$, $-\dfrac{c_+}{[c]}$ decreases, then $\dfrac{u_+}{u_-}$ increases with respect to c_-. □

In the case of two active components we have the following result:

Lemma 5.5 *If $q_1^{''} \le 0 \le q_2^{''}$ and if f is convex then $\dfrac{\partial S}{\partial c_-}(c_+, c_-) \ge 0$.*

Proof: let be c between c_+ and c_-. From Lemma 5.2 we have:

$$u(c) = \frac{f(c_+) - f(c)}{c_+ - c} + 1 + h(c_+) > 0, \qquad v(c) = \frac{f(c_+) - f(c)}{c_+ - c} + 1 + h(c) > 0.$$

We rewrite S using the functions u, v. With Lemma 5.1 we get immediately:

$$\begin{aligned} S(c_+, c_-) &= \ln\left(\frac{[f]/[c] + 1 + h_+}{[f]/[c] + 1 + h_-}\right) \\ &= \ln\left(\frac{u(c_-)}{v(c_-)}\right). \end{aligned}$$

The function f is convex, so u is increasing. ¿From equality $f(c) = q_1(c) - ch(c)$ we have

$$\begin{aligned} (v(c) - 1)(c_+ - c) &= q_1(c_+) - c_+h(c_+) - q_1(c) + ch(c) + h(c)(c_+ - c) \\ &= q_1(c_+) - q_1(c) - c_+(h(c_+) - h(c)). \end{aligned}$$

Recall that $h(c) = q_1(c) + q_2(c)$, so we have:

$$\begin{aligned} (v(c) - 1)(c_+ - c) &= q_1(c_+) - q_1(c) - c_+(q_1(c_+) + q_2(c_+) - q_1(c) - q_2(c)) \\ &= (1 - c_+)(q_1(c_+) - q_1(c)) - c_+(q_2(c_+) - q_2(c)). \end{aligned}$$

Finally, $v(c) - 1 = (1 - c_+)\dfrac{q_1(c_+) - q_1(c)}{c_+ - c} - c_+\dfrac{q_2(c_+) - q_2(c)}{c_+ - c}$ with $0 \le c_+ \le 1$. Now, q_1 is concave and q_2 is convex, so v is decreasing. Finally, $\dfrac{u}{v}$ is increasing and $\dfrac{\partial S}{\partial c_-} \ge 0$. □

6 Interactions estimates

In this section we study the evolution of the total variation of $L = ln(u)$, denoted TVL, through waves interactions. It is a key point to obtain some BV bounds and a special structure for velocity.

Let us denote (c_0, L_0), (c_1, L_1), (c_2, L_2), three constant states such that:

- the Riemann problem with (c_0, L_0) for the left state and (c_1, L_1) for the right state is solved by a simple wave $\mathcal{W}_1$,
- the Riemann problem with (c_1, L_1) for the left state and (c_2, L_2) for the right state is solved by a simple wave $\mathcal{W}_2$,
- $\mathcal{W}_1$ and $\mathcal{W}_2$ interact.

Just after the interaction we have two outgoing waves $\mathcal{W}_1^*$, $\mathcal{W}_2^*$, and the intermediary constant state (c_1^*, L_1^*). We denote by TVL the total variation of $\ln u$ just before interaction:

$$TVL = |L_0 - L_1| + |L_1 - L_2|.$$

We denote by TVL^* the total variation of $\ln u$ just *after* the interaction:

$$TVL^* = |L_0 - L_1^*| + |L_1^* - L_2|.$$

We use similar notation for the concentration.
Denote by α_- the negative part of α: $\alpha_- = \max(0, -\alpha) = -\min(0, \alpha) \geq 0$.
We have the following key estimates:

Theorem 6.1 (Variation on $TV \ln u$ and TVc through two waves interaction) *Assume (26). Then there exists $\Gamma > 0$, a true constant such that:*

$$\begin{aligned} TVL^* &\leq TVL + \Gamma\,|c_0 - c_1|\,|c_1 - c_2|, & (31)\\ TVc^* &\leq TVc. & (32)\end{aligned}$$

Furthermore, if (27) is also satisfied then:

$$TVL^* \leq TVL + \Gamma(c_1 - c_0)_- \times (c_2 - c_1)_-, \qquad (33)$$

in addition, if S, from (28), satisfies the following triangular inequality:

$$S(c_2, c_0) \leq S(c_2, c_1) + S(c_1, c_0)$$

when $c_0 > c_1 > c_2$, then

$$TVL^* \leq TVL. \qquad (34)$$

Inequality (32) means that the total variation of c does not increase and Inequality (33) means that the total variation of $\ln u$ does not increase after a wave interaction except when two shocks interact. In this last case the increase of $TV \ln u$ is quadratic with respect to the concentration variation.

Such estimates are only valid when f has no inflexion point. Else, λ-wave curves are only Lipschitz and we loose the quadratic control for the total variation of L.

Proof of Inequality (32): the decay of the total variation of the concentration is straightforward since c is constant through a contact discontinuity, i.e. $c_1^* = c_0$:
$TVc^* = |c_2 - c_1^*| + |c_1^* - c_0| = |c_2 - c_0| \leq |c_0 - c_1| + |c_1 - c_2| = TVc.$ □

Proof of Inequality (31): this proof is much more complicated. We only assume (26). The proof is a consequence of the following lemmas.

Lemma 6.1 *If a λ-wave interacts with a contact discontinuity then we have $TVL^* = TVL$.*

Proof: it is the simplest case. We have $c_1 = c_2$ from the contact discontinuity, so, with T defined in (28), $L_1 - L_0 = T(c_1, c_0) = T(c_2, c_0)$ and, since $c_1^* = c_0$, we have $L_2 - L_1^* = T(c_2, c_1^*) = T(c_2, c_0)$. Then

$$L_2 - L_1^* = L_1 - L_0,$$

which implies $L_2 - L_1 = L_1^* - L_0$ and $TVL^* = TVL$. $\square$

Lemma 6.2 *There exists a constant $\Gamma > 0$ such that, for all $c_0, c_1, c_2 \in [0,1]$:*

$$|T(c_2, c_0) - T(c_2, c_1) - T(c_1, c_0)| \,|\leq \Gamma \mid c_2 - c_1 \,||\, c_1 - c_0|.$$

Proof: we define R by $R(\alpha, \beta) = T(c_2, c_0) - T(c_2, c_1) - T(c_1, c_0)$. We have to prove that $R(\alpha, \beta) = \mathcal{O}(\alpha\beta)$, where $\alpha = c_1 - c_0$, $\beta = c_1 - c_2$. We denote $c = c_2$, $b = c_1$, $a = c_0$. We have $T \in \mathcal{C}^3([0,1], \mathbb{R})$ since λ is genuinely nonlinear and $T(b, b) = 0$. We apply the Taylor's formula:

$$\begin{aligned}
T(c,a) &= T(b-\beta, b+\alpha) = T(b,b) - \beta\partial_1 T(b,b) + \alpha\partial_2 T(b,b) \\
&\quad + \int_0^1 (1-t)(\beta^2\partial_1^2 S + \alpha^2\partial_2^2 T - 2\alpha\beta\partial_{12}^2 T)(b-t\beta, b+t\alpha)dt, \\
T(b,a) &= T(b, b+\alpha) = T(b,b) + \alpha\partial_2 T(b,b) + \int_0^1 (1-t)\alpha^2\partial_2^2 T(b, b+t\alpha)dt, \\
T(c,b) &= T(b-\beta, b) = T(b,b) - \beta\partial_1 T(b,b) + \int_0^1 (1-t)\beta^2\partial_1^2 T(b-t\beta, b)dt, \\
R(\alpha,\beta) &= T(c,a) - T(c,b) - T(b,a) \\
&= -T(b,b) + \int_0^1 (1-t)(\beta^2(\partial_1^2 T(b-t\beta, b+t\alpha) - \partial_1^2 T(b-t\beta, b)) + \\
&\quad \alpha^2(\partial_2^2 T(b-t\beta, b+t\alpha) - \partial_2^2 T(b, b+t\alpha)) - 2\alpha\beta\partial_1\partial_2 T(b-t\beta, b+t\alpha))dt.
\end{aligned}$$

Since

$$\begin{aligned}
\partial_1^2 T(b-t\beta, b+t\alpha) - \partial_1^2 T(b-t\beta, b) &= \mathcal{O}(t\alpha) = \mathcal{O}(\alpha), \\
\partial_2^2 T(b-t\beta, b+t\alpha) - \partial_2^2 T(b, b+t\alpha) &= \mathcal{O}(t\beta) = \mathcal{O}(\beta), \\
\partial_1\partial_2 T(b-t\beta, b+t\alpha) &= \mathcal{O}(1),
\end{aligned}$$

we conclude that $R(\alpha, \beta) = \mathcal{O}(\beta^2\alpha + \alpha^2\beta + \alpha\beta) = \mathcal{O}(\alpha\beta)$. $\square$

To conclude the proof of Inequality (31) it suffices to use the next lemma.

Lemma 6.3 *If two λ-waves interact then we have $TVL^* \leq TVL + \Gamma|c_2 - c_1|\,|c_1 - c_0|$.*

Proof: by definition of TVL and TVL^* it suffices to prove that

$$L_1^* = L_0 + \mathcal{O}(|c_2 - c_1|\,|c_1 - c_0|),$$

since $TVL^* = |L_2 - L_1^*| + |L_1^* - L_0| \leq |L_2 - L_0| + 2|L_1^* - L_0| \leq TVL + 2|L_1^* - L_0|$.
Indeed, we have: $L_1 - L_0 = T(c_1, c_0)$, $L_2 - L_1 = T(c_2, c_1)$, $L_2 - L_1^* = T(c_2, c_1^*) = T(c_2, c_0)$. Next: $L_2 - L_0 = T(c_2, c_1) + T(c_1, c_0)$ and then

$$L_1^* - L_0 = T(c_2, c_1) + T(c_1, c_0) - T(c_2, c_0),$$

which allows us to conclude the proof of Lemma 6.3 with Lemma 6.2. $\square$

The proof of Inequality (31) is now complete. $\square$

Proof of Inequalities (33), (34):

we assume again (27) and also (29) to fix the signs. There are more cases to study:

- first, we have yet studied in Lemma 6.1 the interaction of a shock wave or a rarefaction wave (λ-wave) with a contact discontinuity (1-wave): the contact discontinuity is "transparent" since $TVL^* = TVL$ and the concentration variation is also invariant.
- second, we study the interaction of a shock wave with a rarefaction wave (λ-waves with different types): see Lemmas 6.4, 6.5, 6.6 and 6.7. We get $TVL^* < TVL$ and the concentration variation decreases. It is the only case where TVL and TVc decrease.
- finally, we study the interaction of two shock waves. In this situation $TVL^* \geq TVL$ and TVc is invariant.
  Furthermore, if S satisfies some "triangular inequality", we get $TVL^* = TVL$.

In order to simplify the notations we denote by D a contact discontinuity, R a rarefaction wave and S a shock wave. " RD $\rightarrow$ DR " means that a rarefaction wave coming from the left interacts with a contact discontinuity and produces a new left wave, namely a contact discontinuity, and a new right wave, namely a rarefaction.
Since a contact discontinuity has a null speed and a λ-wave has a positive speed, the only cases for $\mathcal{W}_1$, $\mathcal{W}_2$ are: RD, SD, RS, SR and SS.
For the resulting waves $\mathcal{W}_1^*, \mathcal{W}_2^*$, there are 7 cases.
The first two cases RD $\rightarrow$ DR and SD $\rightarrow$ DS have yet been studied in Lemma 6.1.

Lemma 6.4 *In the case RS $\rightarrow$ DR, TVL decreases i.e. $TVL^* < TVL$.*

Proof: at the beginning, we have a rarefaction, then $c_0 < c_1$, $L_0 > L_1$, and a shock, then $c_2 < c_1$, $L_2 > L_1$. After the interaction, we have a contact discontinuity, then $c_0 = c_1^*$, and a rarefaction, then $c_1^* < c_2$, $L_1^* > L_2$. Finally, we have $c_0 = c_1^* < c_2 < c_1$ then $g(c_0) = g(c_1^*) \leq g(c_2) \leq g(c_1)$. We can write

$$\begin{aligned} TVL &= \mid L_0 - L_1 \mid + \mid L_1 - L_2 \mid = L_0 - L_1 + L_2 - L_1, \\ TVL^* &= \mid L_0 - L_1^* \mid + \mid L_2 - L_1^* \mid = \mid L_0 - L_1^* \mid + L_1^* - L_2. \end{aligned}$$

There are two cases:

- the simplest is $L_0 > L_1^*$, then $TVL^* = L_0 - L_1^* + L_1^* - L_2 = L_0 - L_2 < L_0 - L_1 < TVL$,
- the second case is $L_0 < L_1^*$. Let us define $\tilde{L}_2$ by

$$L_0 - \tilde{L}_2 = L_1^* - L_2,$$

  then $L_0 - \tilde{L}_2 = L_1^* - L_2 = g(c_2) - g(c_1^*) = g(c_2) - g(c_0) \leq g(c_1) - g(c_0) = L_0 - L_1$ because $[L] = -[g]$ for a rarefaction and $c_1^* = c_0$. Since shock curves are decreasing, we know that $\tilde{L}_2 > L_1$, so $TVL^* = L_1^* - L_0 + L_1^* - L_2 = L_2 - \tilde{L}_2 + L_0 - \tilde{L}_2 < L_2 - L_1 + L_0 - L_1 = TVL$. □

Lemma 6.5 *In the case RS $\rightarrow$ DS we get $TVL^* \leq TVL$.*

Proof: this case needs the assumption $\dfrac{\partial S}{\partial c_-} \geq 0$. At the beginning, we have a rarefaction: $c_1 > c_0$ and $L_1 < L_0$ with a shock: $c_2 < c_1$ and $L_2 > L_1$. The state (c_2, L_2) is connected with a shock (c_1^*, L_1^*): $c_2 < c_1^*$ and $L_1^* < L_2$. The state (c_0, L_0) is connected with a contact discontinuity (c_1^*, L_1^*): $c_0 = c_1^*$. Finally, we have $c_2 < c_0 = c_1^* < c_1$. Then $TVL = \mid L_0 - L_1 \mid + \mid L_1 - L_2 \mid = L_0 - L_1 + L_2 - L_1$ and $TVL^* = \mid L_0 - L_1^* \mid + \mid L_2 - L_1^* \mid = L_2 - L_1^* + \mid L_1^* - L_0 \mid$. But, with the assumption, $\dfrac{\partial S}{\partial c_-} \geq 0$, $S(c_2, c_0) = S(c_2, c_1^*) = L_2 - L_1^* < S(c_2, c_1) = L_2 - L_1$ then $L_1^* > L_1$.
There are two cases:

- if $L_0 > L_1^*$ then $TVL^* = L_0 - L_1^* + L_2 - L_1^* < L_0 - L_1 + L_2 - L_1 = TVL$,
- else $L_0 < L_1^*$ then $TVL^* = -L_0 + L_1^* + L_2 - L_1^* = L_2 - L_0 < L_2 - L_1 < TVL$. □

Lemma 6.6 *In the case SR $\rightarrow$ DR we have $TVL^* \leq TVL$.*

Proof: in the beginning, we have a shock who interacts with a rarefaction then $c_1 < c_0$, $L_1 > L_0$ and $c_2 > c_1$, $L_1 > L_2$.
After the interaction, we have a contact discontinuity then $c_0 = c_1^*$ and a rarefaction then $c_2 > c_1^*$ and $L_1^* > L_2$. Finally, we have $c_1 < c_0 = c_1^* < c_2$. Since $g' \geq 0$, we have $g(c_1) \leq g(c_0) \leq g(c_2)$.
For a rarefaction $[L] = -[g]$ then $L_2 - L_1^* = g(c_1^*) - g(c_2) = g(c_0) - g(c_2)$ because $c_1^* = c_0$,
$L_2 - L_1 = g(c_1) - g(c_2) \leq g(c_0) - g(c_2)$ because $c_1 < c_0$ and $g' \geq 0$.
So we have: $L_2 - L_1 \leq g(c_1^*) - g(c_2) = L_2 - L_1^*$ and
$TVL =\mid L_1 - L_0 \mid + \mid L_2 - L_1 \mid = L_1 - L_0 + L_1 - L_2 \geq L_1 - L_2$,
$TVL^* =\mid L_1^* - L_0 \mid + \mid L_2 - L_1^* \mid = \mid L_1^* - L_0 \mid + L_1^* - L_2$.
There are two cases:

- the first is $L_1^* > L_0$ then $TVL^* = L_1^* - L_0 - L_2 + L_1^* = 2L_1^* - L_0 - L_2 = -(L_2 - L_1^*) + L_1^* - L_0 < -(L_2 - L_1) + L_1^* - L_2 + L_2 - L_0 < -L_2 + L_1 - L_2 + L_1 + L_2 - L_0 = 2L_1 - L_2 - L_0 = TVL$,
- the second case is $L_1^* < L_0$ then $TVL^* = -L_1^* + L_0 - L_2 + L_1^* = L_0 - L_2 \leq L_1 - L_2 \leq TVL$.

□

Lemma 6.7 *In the case SR → DS, TVL decreases i.e. $TVL^* \leq TVL$.*

This situation is illustrated in Fig. 1.

Proof: it is the most difficult case. At the beginning, we have a shock then $c_1 < c_0$ and $L_1 > L_0$. The shock interacts with a rarefaction then $c_2 > c_1$ and $L_2 < L_1$.
We then have $TVL =\mid L_1 - L_0 \mid + \mid L_2 - L_1 \mid = L_1 - L_0 + L_1 - L_2$.
The state (c_2, L_2) is connected to (c_1^*, L_1^*) by a shock then $c_2 < c_1^*$ and $L_1^* < L_2$.
The state (c_0, L_0) is connected to (c_1^*, L_1^*) by a contact discontinuity then $c_0 = c_1^*$.
Finally, we have $c_1 < c_2 < c_1^* = c_0$, $S(c_1, c_0) = S_{10} > S(c_2, c_0) = S_{20} = S(c_2, c_1^*) = L_2 - L_1^*$,
$L_1 - L_0 = S_{10} > S_{20} = L_2 - L_1^*$, because $\dfrac{\partial S}{\partial c_+} < 0$.
There are two cases:

- if $L_0 < L_1^*$ (see Fig. 2, left) then $L_2 < L_1$ and
$$TVL^* =\mid L_1^* - L_0 \mid + \mid L_2 - L_1^* \mid = L_1^* - L_0 + L_2 - L_1^* = L_2 - L_0 < L_1 - L_0 < TVL,$$
- if $L_1^* < L_0$ (see Fig. 2, right) then we define $\tilde{L}_2$ by $\tilde{L}_2 - L_0 = S_{20} = L_2 - L_1^* < S_{10} = L_1 - L_0$ and $TVL^* =\mid L_1^* - L_0 \mid + \mid L_2 - L_1^* \mid = L_0 - L_1^* + L_2 - L_1^* = \tilde{L}_2 - L_2 + S_{20} < L_1 - L_0 + L_1 - L_0 = TVL$.

□

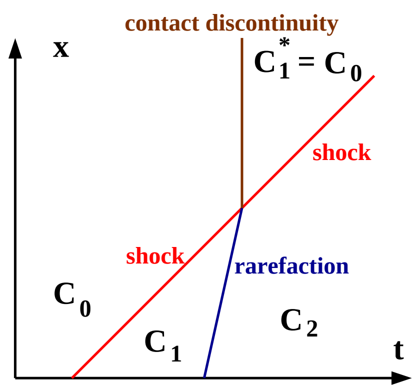

Figure 1: case SR → DS.

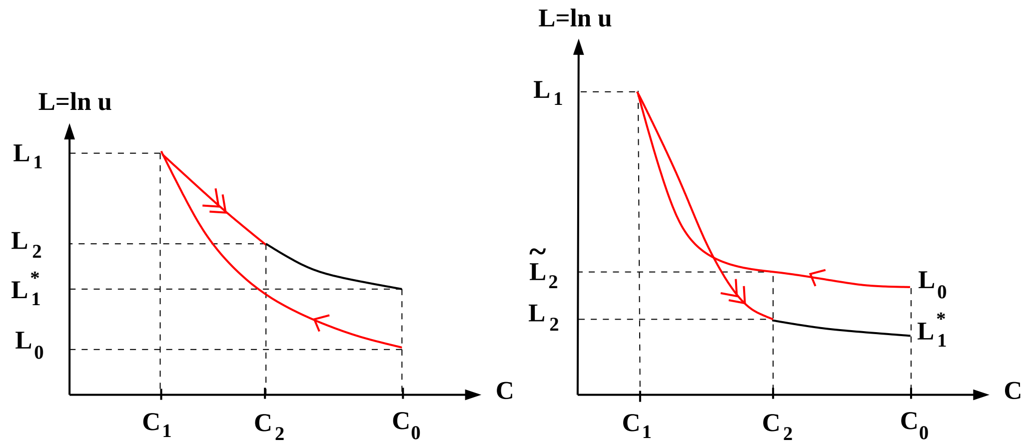

Figure 2: SR → DS: first case, left, and second case, right.

The following case is the only one where TVL increases, except if S satisfies a "triangular inequality".

Lemma 6.8 *In the case $SS \to DS$ we have*

$$TVL^* = TVL + 2\max(S_{20} - S_{21} - S_{10}, 0) = TVL + 2\max(L_0 - L_1^*, 0) \geq TVL.$$

Proof: at the beginning, we have a shock: $c_1 < c_0$ and $L_1 > L_0$. It interacts with an another shock: $c_2 < c_1$ and $L_2 > L_1$.
The state (c_2, L_2) is connected to (c_1^*, L_1^*) by a shock then $c_2 < c_1^*$ and $L_1^* < L_2$.
The state (c_0, L_0) is connected with with (c_1^*, L_1^*) by a contact discontinuity then $c_0 = c_1^*$.
Finally, we have $c_2 < c_1 < c_0 = c_1^*$ and $L_0 < L_1 < L_2$.
With $L_2 - L_1 = S_{21} > 0$, $L_1 - L_0 = S_{10} > 0$, $L_2 - L_1^* = S_{20} > 0$, we have:
$TVL =\mid L_2 - L_1 \mid + \mid L_1 - L_0 \mid = L_2 - L_1 + L_1 - L_0 = S_{21} + S_{10}$,
$TVL^* =\mid L_2 - L_1^* \mid + \mid L_1^* - L_0 \mid = \mid S_{20} \mid + \mid L_1^* - L_2 + L_2 - L_0 \mid = S_{20} + \mid -S_{20} + L_2 - L_0 \mid$.
There are two cases to study:

- if $-S_{20} + L_2 - L_0 \geq 0$ i.e. $S_{20} = L_2 - L_1^* \leq S_{21} + S_{10} = L_2 - L_0$ i.e. $L_0 < L_1^*$ then $TVL^* = S_{20} - S_{20} + L_2 - L_0 = L_2 - L_0 = TVL$,
- else $L_1^* < L_0$ and we have

$$\begin{aligned} TVL^* &= S_{20} + S_{20} - L_2 + L_0 = 2S_{20} - 2L_2 + 2L_0 + L_2 - L_0 \\ &= 2(S_{20} - (L_2 - L_0)) + TVL, = 2(S_{20} - S_{21} - S_{10}) + TVL \\ &= 2(L_0 - L_1^*) + TVL, \end{aligned}$$

which conclude the proof of Lemma 6.8. □

The proof of Theorem 6.1 is now complete. □

7 BV estimates with respect to time for the velocity

In System (1)-(2)-(3), there is no partial derivative with respect to t for u. Nevertheless, the hyperbolicity of this system (with x as the evolution variable) suggests that a BV regularity of the "initial" data u_b for $x = 0$ is propagated. Furthermore, in the case with smooth concentration, the Riemann invariant $u\,G(c)$ suggests that when $\ln u_b$ is only in $L^\infty(0, T)$, we can hope $u(t, x)/u_b(t)$ to be still BV in time for almost all x. We prove that this BV structure of the velocity is still valid with some convexity assumptions, using a Front Tracking Algorithm (FTA). We conjecture

that this structure is still valid for the general case without convexity assumption or, better, with a piecewise genuinely nonlinear eigenvalue $\lambda = H(c)/u$. But, in this last case, the FTA becomes very complicated (see Dafermos' comments in [20]).

7.1 The case $\ln u_b \in BV(0,T)$

We first precise the notations used in the next theorem. We define the function c_I on $(0,T)$ by

$$c_I(s) = \begin{cases} c_0(s) & \text{if } 0 < s < X \\ c_b(-s) & \text{if } 0 < -s < T \end{cases},$$

and we set $TVc_I = TVc_I[-T,X]$.
There exists a positive constant γ such that if (c_-, L_-) is connected to (c_+, L_+) by a λ-wave then $|L_+ - L_-| \le \gamma\,|c_+ - c_-|$. That is an easy consequence of (28). Indeed, it is yet proven in [8], Lemma 3.1, with an inert gas, or in [9], Lemma 4.1, for two active gases.
The constant Γ comes from Theorem 6.1.

Theorem 7.1 [Propagation of BV regularity in time for the velocity]
Assume (26). If $\ln u_b \in BV(0,T)$, if $c_0, c_b \in BV$ and if (u,c) is a weak entropy solution of System (1)-(2)-(3), coming from the Front Tracking Algorithm, then $c \in BV((0,T)\times(0,X))$ and $u \in L^\infty((0,T), BV(0,X)) \cap L^\infty((0,X), BV(0,T))$. More precisely:

$$\begin{aligned}
\max\left(\sup_{0<t<T} TV_x c(t,.)[0,X],\ \sup_{0<x<X} TV_t c(.,x)[0,T]\right) &\le TVc_I,\\
\sup_{0<t<T} TV_x \ln u(t,.)[0,X] &\le TV\ln u_b + \gamma\, TVc_I,\\
\sup_{0<x<X} TV_t \ln u(.,x)[0,T] &\le TV\ln u_b + 2\gamma\, TVc_I + \frac{\Gamma}{2}\,(TVc_I)^2.
\end{aligned}$$

Compared to [8, 9], the new result is that $u(t,x)$ is BV with respect to time if u_b is in $BV(0,T)$ i.e. the last inequality of the Theorem 7.1. With the Godunov scheme used in [8, 9] we do not obtain such time regularity for the velocity. It is the reason why we use the FTA to get more precise estimates. Notice that we consider a local (in time and space) problem for reasons of realism: we could consider a global one as well, i.e. for $(t,x) \in (0,+\infty)^2$.

Proof: The easiest BV estimate on the concentration c after interaction (estimate (32) in Theorem 6.1), which is always valid independently of the velocity u, yields to a control of c in $L^\infty_t BV_x \cap L^\infty_x BV_x$ as in [9], since λ waves always have a positive speed. From Lemma 4.8 of [9] p.80 (ore more simply Lemma 3.1 of [8] p. 557) we get $L^\infty_{t,x} \cap L^\infty_t BV_x$ bounds for the velocity u. It follows, from a natural adaptation of the estimates and compactness argument of the proof of Theorem 5.1 p 563. in [8] or Theorem 6.1 p.83 in [9], that there exists a subsequence which converges to a solution of the initial boundary value problem with the prescribed data c_0, c_b, u_b when δ goes to zero, thanks to the approximate entropy inequality (25). Furthermore, as in [8, 9], we recover strong traces at $t = 0$ and $x = 0$.

Notice that this existence proof is also valid without any BV assumption on the velocity at the boundary: we only need $\ln u_b$ in $L^\infty(0,T)$.

The BV estimate with respect to time for $\ln u$, i.e. the third estimate in the theorem, is a consequence of two following lemmas. □

Let (u,c) be an entropy solution coming from FTA. For $\delta > 0$, representing the distance from the boundary $x = 0$ or $t = 0$, let us define:

$$\begin{aligned}
L(s,\delta) &= \begin{cases} \ln u(t=|s|, x=\delta) & \text{if } -T < s < 0 \\ \ln u(t=\delta, x=s) & \text{if } 0 < s < X \end{cases},\\
TVL(0) &= \limsup_{\delta\to 0} TVL(.,\delta)[-T,X].
\end{aligned}$$

For piecewise data, $TVL(0)$ is the total variation of $\ln u$ just before the first interaction.

Lemma 7.1 *Before wave-interactions we have $TVL(0) \leq TV \ln u_b + 2\gamma TV c_I$.*

Proof: it suffices to prove this inequality for a piecewise constant approximate solution issued from the FTA. We discretize $[0,T]$ and $[0,X]$ as follows:
$T = s_1 > s_2 \cdots > s_m > s_{m+1} = 0 < s_{m+2} < \cdots < s_N = X$.
For $i = 1, \cdots, m$ let us define the following piecewise approximations of c and $\ln u$:

$$c_i = \frac{1}{s_i - s_{i+1}} \int_{s_{i+1}}^{s_i} c_b(t)dt, \qquad L_i = \frac{1}{s_i - s_{i+1}} \int_{s_{i+1}}^{s_i} \ln(u_b(t))dt.$$

Since $t = 0$ is a characteristic boundary we define only c_i for $i = m+1, \cdots, N-1$ by:

$$c_i = \frac{1}{s_i - s_{i+1}} \int_{s_{i+1}}^{s_i} c_0(x)dx.$$

For $i < m$ we solve the i^{th} Riemann Problem with left state (c_i, Li) and right state (c_{i+1}, L_{i+1}) and we denote by c_i^*, L_i^* the intermediary state. Indeed $c_i^* = c_{i+1}$ since c is constant through a contact discontinuity. From Lemma 3.1 p. 557 of [8] (or Lemma 4.1 p.78-79 of [9] for two active gases) we know that:

$$|L_i - L_i^*| \quad \leq \quad \gamma\,|c_i - c_i^*| = \gamma\,|c_i - c_{i+1}|.$$

We now estimate the total variation of $\ln u$ for the i^{th} Riemann problem:

$$\begin{aligned} |L_i - L_i^*| + |L_i^* - L_{i+1}| &\leq |L_i - L_i^*| + (|L_i^* - L_i| + |L_i - L_{i+1}|) \\ &\leq 2\gamma\,|c_i - c_{i+1}| + |L_i - L_{i+1}|. \end{aligned}$$

Now, we look at the corner $t = 0$, $x = 0$ and $i = m$. There is only a λ-wave since the boundary is characteristic. With the left state (c_m, L_m) and only (c_{m+1}) for the right state, the resolution of the Riemann problem gives us a new constant value for $\ln u$, namely $L_{m+1} = L_m^*$. We have again the estimate $|L_m - L_m^*| = |L_m - L_{m+1}| \leq \gamma\,|c_m - c_{m+1}|$. So for $i = m+1, m+2, \cdots, N-1$ we define L_i solving the characteristic Riemann problems with the estimate:

$$|L_i - L_{i+1}| \leq \quad \gamma|c_i - c_{i+1}|.$$

Summing up with respect to i, we obtain the total variation on L just before the first wave interaction:

$$\begin{aligned} TVL &\leq \sum_{i<m} (2\gamma\,|c_i - c_{i+1}| + |L_i - L_{i+1}|) + \sum_{i \geq m} \gamma\,|c_i - c_{i+1}| \\ &\leq TV \ln u_b + 2\gamma TV c_I. \end{aligned}$$

□

Lemma 7.2 *We have the following estimate: $TVL \leq TVL(0) + \frac{\Gamma}{2}(TV c_I)^2$.*

Proof: we prove this estimate for any constant piecewise approximation built from the FTA. The same estimate is still true passing to the limit.

First, we enumerate the absolute value of the jump concentration initial-boundary value from the left to the right:

$$\alpha_i = c_i - c_{i-1} \qquad i = 1, \cdots, N.$$

Notice that we have $N+1$ constant states for the initial-boundary data: $(c_0, L_0), \cdots, (c_N, L_N)$.

From Theorem 6.1, the increase of the total variation of $\ln u$ is governed by following inequality $TVL^* \leq TVL + \Gamma|\alpha_{i-1}||\alpha_i|$ if the wave number $i-1$ interacts with the wave number i. Since c is constant through a contact discontinuity (c is a 2-Riemann invariant) and the jump of c adds up if two λ-waves interact, we consider only interaction between λ-waves. Indeed we neglect that

interaction with rarefaction has the tendency to reduce TVL.
We measure the strength of λ-wave with the jump of c through the wave. We have positive or negative sign whether we have a rarefaction wave or a shock wave.
Let $\left(\alpha_i^k\right)_{1\le i\le N-k}$ be the strength of the λ-wave number i (labeled from the left to the right) after the interaction number k. We have $\alpha_i^0 = \alpha_i$ and denote by j^k the index such that the interaction number k occurs with the λ-wave number j^k and j^k+1 where $1 < j_k \le N-k$. For $1 \le i < N-k$, the strengths of λ-waves after the interaction number $k > 0$ are given by:

$$\alpha_i^k = \begin{cases} \alpha_i^{k-1} & \text{if } i < j^k \\ \alpha_i^{k-1} + \alpha_{i+1}^{k-1} & \text{if } i = j^k \\ \alpha_{i+1}^{k-1} & \text{if } i > j^k \end{cases},$$

and the increasing of TVL is less or equal than ΓS^k where, from Theorem 6.1,

$$S^0 = 0, \qquad S^k = S^{k-1} + |\alpha_i^{k-1}||\alpha_{i+1}^{k-1}|.$$

Let us define the integers l_i^k as follows:
$l_i^0 = i$ and at each interaction

$$l_i^k = \begin{cases} l_i^{k-1} & if \quad i < j^k, \\ l_{i+1}^{k-1} & if \quad i = j^k, ..., N-k+1. \end{cases}$$

Notice that after each interactions with two λ-waves, there is only one outgoing λ-wave. Thus, the number of λ-waves decreases at each interactions, which proves again (see [21]) that the number of interactions is finite and the FTA is well posed.

By induction, we see that: $\alpha_i^k = \displaystyle\sum_{l_i^k \le l < l_{i+1}^k} \alpha_l$ where $l_1^k = 1 < l_2^k < \cdots < l_{N-k+1}^k = N-k+1$, $l_i^0 = i$ and l_i^k is non decreasing with respect to k. Now, from the definition of S^k, we can deduce that:

$$S^k = S^{k-1} + \sum_{(i,j)\in J^k} |\alpha_i||\alpha_j|, \tag{35}$$

where $J^k = \{(i,j);\ l_{j^k}^{k-1} \le i < l_{j^k+1}^{k-1} \le j < l_{j^k+2}^{k-1}\}$.
Let us check that:

$$S^k = \sum_{(i,j)\in I^k} |\alpha_i||\alpha_j|, \tag{36}$$

where $\emptyset = I^0 \subset I^1 \subset \cdots \subset I^{k-1} \subset I^k \subset \cdots \subset I = \{(i,j);\ 1 \le i < j \le N\}$.

It is true for $k = 0$. It is true for all k if $I^{k-1} \cap J^k = \emptyset$ and then $I^k = I^{k-1} \cup J^k$. The point is only to prove that $I^{k-1} \cap J^k = \emptyset$. Terms $|\alpha_i||\alpha_j|$ in the last sum of (35) have indexes i and j which appear in two consecutive intervals, i.e. $l_{j^k}^{k-1} \le i < l_{j^k+1}^{k-1} \le j < l_{j^k+2}^{k-1}$ and after, for $i = j^k$, $l_i^k = l_i^{k-1}$ and $l_{i+1}^k = l_{i+2}^{k-1}$. So i and j live in the same interval and then terms $|\alpha_i||\alpha_j|$ cannot appear again in S^{k+1}, S^{k+2}, ..., since such intervals are not decreasing.
The same is true for all indexes in I^k. They can appear at most one time in S^k. We then have $I^{k-1} \cap J^k = \emptyset$ and (36) is true.

We easily estimate S^k, which concludes the proof:

$$S^k \le \sum_{(i,j)\in I} |\alpha_i||\alpha_j| \le \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N |\alpha_i||\alpha_j| = \frac{1}{2}\left(\sum_{i=1}^N |\alpha_i|\right)^2 \le \frac{1}{2}\left(TVc_I\right)^2.$$

$\square$

7.2 The case $\ln u_b \in L^\infty(0,T)$

For $\ln u_b \in L^\infty$ and $c_0, c_b \in BV$ we get a BV structure for the velocity.

Theorem 7.2 **[BV structure for the velocity]** *We assume (26).*
If $\ln u_b \in L^\infty(0,T)$, if c_0, $c_b \in BV$ and if (c,u) is a weak entropy solution issued from the FTA, then

$$\max\left(\sup_{0<t<T} TV_x c(t,.)[0,X], \sup_{0<x<X} TV_t c(.,x)[0,T]\right) \leq TV c_I$$

and there exists a function v and constants γ, $\Gamma > 0$ such that $u(t,x) = u_b(t) \times v(t,x)$ with

$$\ln v \in \{L^\infty((0,X), BV(0,T)) \cap L^\infty((0,T), BV(0,X))\} \subset BV((0,T) \times (0,X)),$$

$$\begin{array}{rcl} \sup\limits_{0<t<T} TV_x \ln v(t,.)[0,X] & \leq & \gamma\, TV c_I, \\ \sup\limits_{0<x<X} TV_t \ln v(.,x)[0,T] & \leq & 2\,\gamma\, TV c_I + \dfrac{\Gamma}{2}\left(TV c_I\right)^2. \end{array}$$

The new result in this theorem is that $\dfrac{u(t,x)}{u_b(t)}$ is BV with respect to time, although u_b is not assumed to be BV, but just in L^∞. The other regularity properties have yet been proved in [8, 9].

Proof: the first estimates for c are easily obtained as in Theorem 7.1 since the total variation of the concentration does not increase after an interaction. The existence proof of such entropy solution follows the beginning of the proof of Theorem 7.1 which is a natural adaptation of the existence proof from [8, 9] with only L^∞ velocity.
We now study the new BV estimates for v. We can define v by the relation $u(t,x) = u_b(t)v(t,x)$ because $u_b > 0$. Let be $M = \ln v$ and $M_b = \ln v(\cdot, x=0)$. The initial total variation of M on $x = 0$ is $TVM_b = 0$ since $v(t, x=0) = 1$.
We approach u_b with a piecewise constant data (thus in BV) and we show that the BV estimate for M is independent of u_b. Notice the fundamental relation:

$$[L] = \ln u_+ - \ln u_- = \ln(u_b(t)\, v_+) - \ln(u_b(t)\, v_-) = \ln v_+ - \ln v_- = [M].$$

The equality $[L] = [M]$ implies that the λ-waves (28) are the same in coordinates (c, L) and (c, M). Then, Theorem 6.1 is still valid replacing L by M. We then can repeat the proof of Theorem 7.1 to get BV estimates for v. □

8 Weak limit for velocity with BV concentration

When c is only in BV, we cannot reduce System (4) to a scalar conservation law for c as in section 3. Indeed, since the shock speeds depend on the velocity, we have a true 2×2 hyperbolic system. Nevertheless we can state following stability result.

Theorem 8.1 (Stability with respect to weak limit for the velocity)
Let $(\ln(u_b^\varepsilon))_{0<\varepsilon<1}$ be a bounded sequence in $L^\infty(0,T)$, such that

$$u_b^\varepsilon \rightharpoonup \overline{u}_b \text{ in } L^\infty(0,T) \text{ weak } *.$$

Let be $c_0 \in BV((0,X),[0,1])$ and $c_b \in BV((0,T),[0,1])$. Let $(c^\varepsilon, u^\varepsilon)$ be a weak entropy solution of System (4) on $(0,T) \times (0,X)$ issuing from the FTA with initial and boundary values:

$$\left\{ \begin{array}{rcl} c^\varepsilon(0,x) & = & c_0(x), \quad X > x > 0, \\ c^\varepsilon(t,0) & = & c_b(t), \quad T > t > 0, \\ u^\varepsilon(t,0) & = & u_b^\varepsilon(t), \quad T > t > 0. \end{array} \right.$$

Then, there exists $(u(t,x), c(t,x))$, weak entropy solution of System (4) supplemented by initial and boundary values:

$$\begin{cases} c(0,x) &= c_0(x), \quad x > 0, \\ c(t,0) &= c_b(t), \quad t > 0, \\ u(t,0) &= \overline{u}_b(t), \quad t > 0, \end{cases}$$

such that, when ε goes to 0 and up to a subsequence:

$$\begin{aligned} c^\varepsilon(t,x) &\to c(t,x) \text{ strongly in } L^1([0,T]\times[0,X]), \\ u^\varepsilon(t,x) &\rightharpoonup u(t,x) \text{ weakly in } L^\infty([0,T]\times[0,X]) \text{ weak *}, \\ u^\varepsilon(t,x) &= u_b^\varepsilon(t) \times v(t,x) + o(1) \text{ strongly in } L^1([0,T]\times[0,X]), \text{ where } v(t,x) = \frac{u(t,x)}{\overline{u}_b(t)}. \end{aligned}$$

For the convergence of the whole sequence we need the uniqueness of the entropy solution for initial-boundary value problem: (4), (5).

Proof: from Theorem 7.2 we know that $u^\varepsilon(t,x) = u_b^\varepsilon(t) v^\varepsilon(t,x)$ where the sequences $(\ln v^\varepsilon)_{0<\varepsilon}$ and $(c^\varepsilon)_{0<\varepsilon}$ are uniformly bounded in $BV((0,T)\times(0,X))$. Then, up to a subsequence, we have the following strong convergence in $L^1((0,T)\times(0,X))$: $v^\varepsilon \to v$, $c^\varepsilon \to c$.

$(c^\varepsilon, u^\varepsilon)$ is a weak entropy solution for (4) means for all ψ such that $\psi" \geq 0$ and Q such that $Q' = h'\psi + H\psi'$ we have in distribution sense: $\partial_x(u^\varepsilon(t,x)\psi(c^\varepsilon)) + \partial_t Q(c^\varepsilon) \leq 0$,which is rewritten as follows: $\partial_x(u_b^\varepsilon(t) v^\varepsilon(t,x)\psi(c^\varepsilon)) + \partial_t Q(c^\varepsilon) \leq 0$. Passing again to the weak-limit against a strong limit we get: $\partial_x(\overline{u}_b(t)v(t,x)\psi(c)) + \partial_t Q(c) \leq 0$. i.e. $(c, u = \overline{u}_b \times v)$ is a weak entropy solution for System (4). We also can pass to the limit on initial-boundary data.

Since there exists δ such that $0 < \delta < u_b^\varepsilon < \delta^{-1}$, $v^\varepsilon(t,x) \to v(t,x)$ means $u^\varepsilon(t,x)/u_b^\varepsilon(t) - v(t,x) \to 0$ and also means $u^\varepsilon(t,x) - u_b^\varepsilon(t) \times v(t,x) \to 0$, which concludes the proof. □

An example of high oscillations for velocity: as an example of weak limit we consider the case of high oscillations for velocity on the boundary.

Let be $u_b(t,\theta) \in L^\infty((0,T), C^0(\mathbb{R}/\mathbb{Z},\mathbb{R}))$, $\overline{u}_b(t) = \displaystyle\int_0^1 u_b(t,\theta)d\theta$ and assume $\inf u_b > 0$. With $u_b^\varepsilon(t) = u_b\left(t, \dfrac{t}{\varepsilon}\right)$ and the same notations as in Theorem 8.1 we have:

- first, oscillations do not affect the behavior of the concentration since (c^ε) converges strongly in L^1 towards c and the limiting system depends only on the average $\overline{u}_b$ and not on oscillations;
- second, (u^ε) converges weakly towards $\overline{u}_b(t) \times v(t,x)$ and we have a strong profile for u^ε:

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(t,x) - U\left(t,x,\frac{t}{\varepsilon}\right)\right\|_{L^1((0,T)\times(0,X))} = 0, \text{ where } U(t,x,\theta) = u_b(t,\theta) \times v(t,x).$$

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